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*EVIDENCE FOR AN INTERRELATION IN THE METABOLISM OF LYSINE, ARGININE AND PYRIMIDINES IN NEUROSPORA**

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The behavior of a strain of *Neurospora crassa* carrying the pyrimidineless gene *3a* (37301) and *s*, a second mutant gene which suppresses the effect of *3a*, has been described briefly in an earlier publication.[1] A strain carrying *3a* alone has an absolute requirement for pyrimidine, while *3a-s* reaches about one-half the maximum growth of the wild type on the unsupplemented basal medium.[2] However, if arginine is present in this medium the pyrimidine requirement is again manifested. The effect of arginine may be suppressed in turn by the addition of lysine. This response to arginine and lysine suggested the possibility that one or both of these amino acids are involved in the biosynthesis of pyrimidines. More recent investigations have provided evidence that a relationship does exist, but its nature is not yet understood.

Of the two lines of evidence which indicate this interrelation perhaps the most convincing is the fact that two of four non-allelic lysineless mutants[3] accumulate pyrimidine compounds in their culture media. One of the compounds has been isolated and identified as uracil. A second substance, more active biologically, has been shown to be present but has not been isolated. There is some evidence which suggests that this compound is uridine.

A second line of evidence is based on growth responses of multiple mutant strains derived from crosses of lysineless mutants to pyrimidineless *3a-s*. The unexpected behavior of several of these strains has not been satisfactorily interpreted, but, in the opinion of the authors, such an interpretation must involve an interrelation in the metabolism of pyrimidines and lysine, and, possibly of arginine as well.

Accumulation of Pyrimidines by Lysineless Strains.—Four lysineless

mutants, shown by Doermann[3] to be non-allelic, have been tested. The isolation numbers are 33933, 37101, 4545 and 15069; for simplicity they will be called *ly 1*, *2*, *3* and *4*. After the finding by Borsook, *et al.*,[4] that lysine is converted to α amino adipic acid by mammalian tissue slices and homogenates it was shown that this amino acid can be used as a substitute for lysine by strain *ly 1* but not by *ly 2*, *3* and *4*.[5] From this it is inferred that the reaction blocked in strain *ly 1* precedes those blocked in the other strains.

Accumulated pyrimidine compounds have been demonstrated in culture fluid from *ly 3* and *4* but not in that from *ly 1* and *2*. The amount found, in terms of cytidine sulfate activity for the pyrimidineless strain *3a*, varied from 0.1 to 0.2 mg. in 10 ml. of culture fluid. The substances promote growth of three non-allelic pyrimidine-requiring strains, 37301 or *3a*, 263 and 38502.[6] The accumulation was observed in four-day cultures containing 1 mg. of L(+) lysine monohydrochloride in 20 ml. of medium. This quantity of lysine is sufficient to allow half-maximum growth of the lysineless strains. When higher lysine concentrations were used little or no pyrimidine activity was detected.

Of the four lysineless strains, *1* and *3* have been more thoroughly tested. In repeated experiments *ly 3* has not failed to show the accumulation under the conditions given above while *ly 1* has always failed to do so. Moreover, no accumulation by the double mutant *ly 1–ly 3* has been observed. In three experiments with *ly 2* and *ly 4* the former has behaved like *ly 1* and the latter like *ly 3*. Culture fluid from wild type (Abbott 4) and from an adenine-requiring strain 27663-44206 supplied a limiting quantity of adenine, showed no biological activity for the pyrimidineless strains.

Uracil was isolated from a culture of *ly 3* grown four days at 25°C. in a five-gallon bottle containing 15 l. of basal medium supplemented with 750 mg. of L(+) lysine monohydrochloride. The material was first concentrated by adsorption on Norite (2 mg. per ml. of culture fluid) and elution with boiling 10% aniline solution. The active compounds were then precipitated by the addition of excess silver nitrate at about pH 8. Silver was removed as the sulfide, the volume reduced by evaporation and crystalline compound was obtained. After two recrystallizations from hot water the compound melted with decomposition at 335°C. (corr.). Literature values for uracil are 338°C., 335°C. The ultra-violet absorption spectrum was identical to that of uracil and the biological activity for *pyr 3a* was also the same.

It seems clear that all of the biological activity found is not due to uracil. The activity of a number of partially purified preparations exceeded by a considerable margin that of the quantity of uracil which will remain dissolved in the same volume of water. Moreover, solutions of such highly active preparations showed the maximum absorption of ultra-violet light at the same wave-length (259 mμ) in 0.1 *N* NaOH as in 0.1 *N* HCL. This

is characteristic of the spectrum of uridine, while solutions of uracil, on the other hand, show a change from 259 mμ in acid to 283 mμ in alkali.

It was found that a small quantity of cytidine, which has the same biological activity as uridine (about 30 times that of uracil[6]) will greatly enhance the activity of uracil. A mixture of 0.1 mg. of cytidine sulfate and 1.6 mg. of uracil is equivalent in activity to 8 mg. of uracil alone. It seems safe to conclude, therefore, that a pyrimidine derivative with the activity of uridine or cytidine is also accumulated. The ultra-violet absorption spectra mentioned above indicate that it is uridine.

TABLE 1

REQUIREMENTS FOR HALF-MAXIMUM GROWTH OF SINGLE AND MULTIPLE MUTANT STRAINS

STRAIN	REQUIREMENT—MICROMOLS/20 ML. MEDIUM
s	0
pyr 3a	cytidine sulfate 1.6; uracil 54.0
	cytidine sulfate 0.31 + uracil 2.7
pyr 3a-s	0
ly 1	L—lysine 5.5
	DL α amino adipic acid 6.2
ly 1-s	L—lysine 5.5
	DL α amino adipic acid 25.0
	DL α amino adipic acid 6.2 + L—arginine 0.47
	DL α amino adipic acid 6.2 + DL citrulline 5.7
	DL α amino adipic acid 6.2 + DL ornithine 12.0
ly 1-pyr 3a	L—lysine 5.5 + cytidine sulfate 1.6
	DL α amino adipic acid 6.2 + cystidine sulfate 1.6 + L—lysine 0.55
ly 1-pyr 3a-s	L—lysine 5.5
ly 3	L—lysine 5.5
ly 3-s	L—lysine 5.5
ly 3-pyr 3a	L—lysine 5.5 + cytidine sulfate 1.6
ly 3-pyr 3a-s	L—lysine 5.5 + cytidine sulfate 1.6 + L—arginine 2.4

Double and Triple Mutants Involving Ly 1, Pyr 3a and s.—The requirement for lysine due to the mutant gene *ly 1* is not affected by the presence, in the same strain, of *s* or *pyr 3a*, nor is the pyrimidine requirement or *pyr 3a* altered by the presence of *ly 1*. Growth of the triple mutant *ly 1–pyr 3a-s*, when supplied adequate lysine, parallels that of *pyr 3a-s* on the unsupplemented basal medium. However, with respect to their ability to use α amino adipic acid as a substitute for lysine each of the three strains, *ly 1-s*, *ly 1-pyr 3a* and *ly 1-pyr 3a-s* differs from strain *ly 1*.

The quantity of α amino adipic acid required to initiate growth of strain *ly 1-s* is about four times that which is necessary for half-maximum growth of *ly 1*, but if arginine, citrulline or ornithine is added then the response of this strain is very nearly like that of the single mutant. Citrulline is less effective than arginine and ornithine is still less effective; also the effect of the latter is rather erratic. The quantities required are given in table 1.

This response to these amino acids is of particular interest because of their effect upon the lysineless strains and upon *pyr 3a-s*. Their presence in the culture medium inhibits the utilization of lysine by all of the lysineless strains[7] and prevents growth of strain *pyr 3a-s* in the absence of pyrimidine.[1] The relative effectiveness of arginine, citrulline and ornithine is the same in the three cases.

Strain *ly 1-pyr 3a* does not grow when supplied α amino adipic acid with cytidine and cannot be made to do so by the addition of arginine, citrulline or ornithine. However, if a small quantity of lysine is added, about one-tenth the requirement for half-maximum growth, then, in the presence of adequate cytidine, the response of this strain to α amino adipic acid is like that of *ly 1*. Utilization of this compound by *ly 1-s* is apparently not affected by lysine. When supplied lysine in addition to quantities of α amino adipic acid smaller than that required to initiate growth, this strain responds as it does to the same quantity of lysine alone.

Utilization of α amino adipic acid by *ly 1-pyr 3a-s* has not been demonstrated although the following additions have been tried: lysine; arginine plus cytidine; and lysine plus arginine plus cytidine.

No inhibitory effects have been observed of cytidine upon the utilization of α amino adipic acid by strain *ly 1*, nor of this amino acid upon the utilization of cytidine by *pyr 3a*. Growth of strain *pyr 3a-s* is not affected by α amino adipic acid.

Double and Triple Mutants Involving Ly 3, Pyr 3a and s.—Requirements for lysine and cytidine of the double mutants *ly 3-s* and *ly 3-pyr 3a*, like the corresponding combinations with *ly 1*, have not been found to differ from those of the single mutants. Strain *ly 3-pyr 3a-s*, on the other hand, departs markedly from the behavior which would be predicted for it on the basis of characteristics of the single and double mutants involving these three genes. It does not grow when supplied lysine alone, but requires pyrimidine and arginine as well. Moreover, in order to allow maximum growth, lysine and arginine must be supplied in a molar ratio of 2.3 to 1. The requirement for pyrimidine is not surprising since the suppressor does not function if arginine is present unless the lysine concentration is about six times that of arginine. When lysine and arginine are supplied in suitable concentrations then the response of this strain to cytidine is the same as that of *pyr 3a*. Table 1 lists the requirements for half-maximum growth of the various single and multiple mutant strains.

There are two points of similarity between *ly 3-pyr 3a-s* and the argininless mutant 36703. First, neither citrulline nor ornithine will satisfy the arginine requirement.[8] Second, both strains are inhibited by lysine if the molar ratio of lysine concentration to arginine concentration is 3.5 to 1 or greater. For this reason strains of the constitutions *arg-s* and *arg-pyr 3a* were considered of possible interest and were prepared. However, no

departures from predicted behavior have been observed. *Arg-s* responds to arginine and lysine as does *arg* alone, while requirements of *arg-pyr 3a* for arginine and cytidine are like those of *arg* and *pyr 3a*. Combinations involving *arg* and *ly 1* or *ly 3* may prove of interest but they have not yet been obtained.

Verification of Genetic Constitution of Multiple Mutant Strains.—The various double and triple mutants were obtained from crosses of *pyr 3a-s* to *ly 1*, *ly 3* and *arg*. *Ly 1-s*, *ly 3-s* and *arg-s* were selected from asci containing four spores carrying *pyr 3a* alone. The other four spores would, of course, carry *s* along with the third mutant gene involved in the cross. The double mutants of *pyr 3a* with *ly 1*, *ly 3* or *arg* were obtained from asci which gave rise to four wild-type appearing strains and four having the appropriate double requirement, lysine plus pyrimidine or arginine plus pyrimidine. *Ly 1-pyr 3a-s* came from an ascus from which were obtained four wild-type appearing strains and four requiring only lysine. The ascus from which *ly 3-pyr 3a-s* was derived gave rise to four wild-type strains and four which failed to grow when supplied lysine plus cytidine. The identity of these two strains was further checked by recovering from crosses to wild-type strains carrying *pyr 3a-s* and others requiring lysine alone or lysine plus pyrimidine. *Arg-pyr 3a* had, of course, to be crossed to wild type in order to distinguish it from *arg-pyr 3a-s*.

In the case of each multiple mutant described, isolates from several different asci were tested and found to have the same characteristics.

Discussion.—The fact that obstruction of the biosynthesis of lysine at either of two points results in accumulation of pyrimidines, while interruption of the series at two other points does not produce this result, establishes an interdependence in the metabolism of pyrimidine and lysine in Neurospora. Such a relationship is also strongly indicated by the fact that introduction of a genetic block in the series of reactions by which the pyrimidine ring is synthesized interferes with the utilization of a precursor of lysine, α amino adipic acid, by strain *ly 1-pyr 3a*. The behavior of *ly 1-s* and *ly 1-pyr 3a-s*, as well as that of *ly 1-pyr 3a*, suggests that utilization of this compound is dependent upon the *pyr 3a* reaction.

No satisfactory scheme has been devised which will explain how lysine metabolism is related to pyrimidine biosynthesis and in what way arginine is involved.

Summary.—Of four non-allelic lysineless strains two have been shown to accumulate pyrimidines, while two do not. One of the accumulated compounds has been isolated and identified as uracil. Another has been tentatively identified as uridine.

The double and triple mutant combinations of the pyrimidineless gene, *3a* and its suppressor, *s*, with two lysineless strains have been prepared and tested. Several of these show growth responses, not characteristic of the

single mutants involved, which indicate a relationship in the metabolism of pyrimidines, lysine and arginine. Combination of the genes, lysineless *3*, pyrimidineless *3a* and *s* introduces a requirement for arginine. Lysineless strain *1* is able to use α amino adipic acid as effectively as it uses lysine. Combination of this gene with the suppressor gene *s* produces a strain which has the same requirement for lysine as lysineless *1*, but a much higher requirement for α amino adipic acid, unless arginine, citrulline or ornithine is also supplied. By combination of this lysineless gene with pyrimidineless *3a* a strain is produced which cannot use α amino adipic acid unless a small amount of lysine is added.

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*THE CYTOGENETIC EFFECT OF SONIC ENERGY APPLIED SIMULTANEOUSLY WITH X-RAYS**

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It is generally agreed that the number of chromosome breaks initially produced by a given dose of x-rays is independent of conditions at the time of radiation. But the subsequent yield of observable chromosomal aberrations from a given number of initial breaks depends upon the ratio of those breaks which rejoin (and are not detectable) to those which do not rejoin or which rejoin in detectable new associations. The majority of the breaks produced by x-radiation, under normal conditions, seem to rejoin in their original position and do not form observable aberrations.[3]

Any factors which affect this process of rejoining will influence the yield of chromosomal aberrations obtained from a given x-ray dose. Several treatments used in conjunction with x-rays have been shown to increase (or decrease) the yield as compared with that obtained from the same dose

of x-rays alone. The results of some of these treatments are summarized in table 1:

TABLE 1

EFFECT ON X-RAY ABERRATION FREQUENCY

TREATMENT	EFFECT ALONE	BEFORE	DURING	AFTER	REFERENCE
Centrifugation	0	x	+(2/1)*	0	(5)
Temperature (3°/36°)	0	0	+(4/1)	0	(4)
Colchicine	0	−(1/3)	x	x	(2)
Ultra-violet	+	−(1/2.5)	x	−(1/2)	(8)
Infra-red	0	+(2/1)	x	+(2/1)	(9)

* Ratio, x-ray + treatment/x-ray alone.
x = not studied.

In addition to the treatments above, the effect of the stage of division at the time of radiation, genetic differences and differences between different tissues of the same individual have been studied.[6]

Most of these artificial or natural influences (except ultra-violet and infra-red) seem to produce their effect on the yield of chromosomal aberrations by altering somehow the amount or freedom of chromosome movement. An increase in the amount of movement after chromosomes have been broken by x-rays would enable the broken ends from one break to be separated from one another (reducing restitution) and would bring broken ends from different breaks together (increasing new reunions); either would increase the yield of aberrations. These considerations, and the work which had been done with centrifugation, suggested that sonic or supersonic energy would be a means of applying mechanical agitation to cellular structures during radiation, thereby increasing the yield of chromosomal aberrations.

Experimental Methods.—Inflorescences of a clone of *Tradescantia paludosa* Anderson and Woodson were subjected to simultaneous x-ray and sonic treatment. Aceto-carmine smears of the anthers were made four days after treatment, and the frequency of chromosomal aberrations determined at the microspore mitosis. X-rays were delivered at 250 kv. peak, 15 ma., filtered through $^1/_2$ mm. Cu and 1 mm. Al. The average intensity, measured with a Victoreen dosimeter in air at the same position as the buds were when exposed to radiation, was 78 r/minute. A dose of 250 r was given in all experiments.

Sonic treatment was begun with the x-radiation and continued for five minutes after radiation ceased. The sonic treatment was administered by a Raytheon Manufacturing Company Model R-22-3 magnetostriction transducer connected to an electronic oscillator operating at about 9100 cycles/second.[7] The material was treated in a stainless steel cup, the bottom of which was a diaphragm connected directly to a nickel magneto-

striction rod which alternately contracted and expanded in an alternating electromagnetic field. The sound energy generated by the cup diaphragm was transmitted to the Tradescantia buds by water 3.4 cm. deep (30 cc.) in which the inflorescences were immersed, upside down, the center of the buds being 3 mm. beneath the surface of the water through which the sonic vibrations were transmitted. Temperature was kept constant by a continuous flow of tap water (18°C.) through the water jacket of the double-walled sonic cup. Inflorescences were held in place during treatment by inserting the stem base of the inflorescence through a hole in a thin lucite cap fitted over the top of the cup. The machine was always operated at the same voltage level across the transducer, 30 volts, and at the same frequency, about 9100 cycles/second. The sonic cup, with the flower buds held in position in the water, was centered under the x-ray tube; in both the control series (x-rays only) and the treated series (x-ray plus sonic energy), buds were held in the same position in the water inside the sonic cup during treatment. The relative x-ray dose received by the buds in the sonic and control series was the same; however, the absolute dose, due to scattering by the lucite cap, water and steel cup, may not have been the same as the 250 r measured by a dosimeter in the same position in air.

Results.—A series of six experiments was made. Since the results of all are essentially the same, the data have been combined and are presented in table 2.

TABLE 2

YIELD OF FOUR-DAY CHROMOSOMAL ABERRATIONS PER 100 CELLS. 250 R X-RAYS (AV. 78 R/MIN.); AV. 7½ MINUTES SONIC TREATMENT

TREATMENT	NO. OF BUDS	NO. OF CELLS	EXCHANGES* NO.	EXCHANGES* PER 100 CELLS	DELETIONS† NO.	DELETIONS† PER 100 CELLS
X-rays + sonic	21	1481	547	36.9 ± 1.4	751	50.7 ± 1.9
Control (x-ray alone)	8	650	189	29.1 ± 2.0	261	40.2 ± 2.5
Diff. (sonic) − (x-rays) ± S.E. diff.	..	..	...	7.8 ± 2.5	...	10.5 ± 3.2
Ratio: sonic/x-ray	..	..	...	1.27/1	...	1.26/1

* Dicentrics + tricentrics + centric rings.

† Terminal + small isodiametric fragments (acentric rings excluded).

Previous experiments had shown that the sonic treatment alone, as given here, did not cause detectable chromosomal aberrations.† The yield of x-ray induced aberrations has been increased by the application of sonic energy; the differences between the sonic treated and x-ray controls are statistically significant for both the one- and two-hit aberrations (difference > 3 times S.E. diff.).

Discussion.—The increase in aberration yield is somewhat less than that obtained by other treatments, where the ratio of treated/control has been

about 2/1 for centrifugation, and infra-red, about 4/1 for cold treatment. It is believed that the smaller effect found with sonic energy, as administered in these experiments, is due to the fact that it is difficult to deliver a sonic dose intense enough to produce the effect desired without causing a disruption of the cells or entire tissue, making further observation impossible. An intensity of 30 volts across the transducer was empirically determined as the maximum intensity which could be used, and still have the buds and anthers survive intact. In these experiments, cells have been observed as much as five days after treatment in which the entire nucleus had been disrupted into numerous small droplets scattered throughout the cell, in other cases the nucleus had been condensed into a solid, amorphous pycnotic globule. It is interesting that some cells, five days after treatment, though still intact and at the same stage of cellular development as normal cells, had no vestige of the nucleus or nuclear fragments remaining, though the cytoplasm was uniformly darker stained than nearby normal cells. Apparently the nucleus was almost completely dispersed throughout the cells, yet five days of development and growth of the microspore ensued in such cells with a dispersion rather than an organization of genic material. There can be no doubt, however, that the transmission of sonic energy does cause movement of cellular constituents without disrupting the cell; such effects have been described in early biological studies made with sonics and supersonics.[1] Further, an x-ray induced chromosome fragment, because of its smaller mass and surface area, should have a smaller viscous drag and therefore be displaced by sonic vibration more than the heavier and larger chromosome from which it came. This may be observed qualitatively for isodiametric deletions ("minutes" or "dots") which, following x-ray treatment alone, are generally associated quite closely with the chromosomes; after combined x-ray and sonic treatment, the deletions are often completely dissociated from the chromosomes.

It is thought that by refinements in the method of treatment, and perhaps by using ultrasonic energy of much shorter wave-length than used here (9100 cycles/second = about 16 cm. wave-length in water) greatly enhanced yields of aberrations may be obtained; if the problem of inducing considerable vibrational movement of cellular constituents without causing disruption of the cell wall can be solved, this should be possible. Ultrasonics with wave-lengths of the order of magnitude of cell dimensions may be the way of doing this.

Summary.—The cytogenetic effects of simultaneous treatment of Tradescantia inflorescences with x-rays (250 r) and sonic energy (9100 cycles/second) are described. Simultaneous treatment with sonic energy and x-rays increases the yield of x-ray induced chromosomal aberrations about 1.3 times the yield obtained with the same amount of x-rays alone.

The sonic treatment alone did not cause aberrations. The increased yield is probably due to an increased movement of chromosomes and chromosome fragments caused by the sonic treatment, resulting in a decrease in the amount of restitution and an increase in the amount of detectable new reunions between the broken ends of chromosomes.

† A recent experiment[10] reports the production of chromosomal aberrations, mutations and other effects in Allium root tips, Helianthus stem apices, and Drosophila by treatment with ultrasonics alone. However, the treatment was quite different in frequency (400,000 cycles/second) and in other factors from the sonic exposures reported here.

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RAYLEIGH-RITZ AND A. WEINSTEIN METHODS FOR APPROXIMATION OF EIGENVALUES. I. OPERATORS IN A HILBERT SPACE*

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The well-known method of Rayleigh-Ritz[1] permits us to find upper bounds for the eigenvalues of a differential operator. In the late thirties, A. Weinstein introduced a new method, in some cases important for applications, giving lower bounds for the eigenvalues.[2]

We were able to extend Weinstein's method, making it more precise and simplifying it at the same time by the use of the Hilbert space theory.

This method is seen then as a counterpart of the Rayleigh-Ritz method, or more precisely, of a generalized Rayleigh-Ritz method for which similar developments to Weinstein's method can be established.

In the present paper we develop the theory for completely continuous, symmetric operators in a Hilbert space. In another paper we shall study the applications of this theory to differential self-adjoint operators.

1. Consider in a Hilbert space $\mathfrak{H}$, with the scalar product (x, y), an operator H, symmetric and completely continuous.

For any linear closed subspace $\mathfrak{L} \subset \mathfrak{H}$ we denote by P the corresponding projection.

The operator $L = PH$, considered in the subspace $\mathfrak{L}$, will be called *the part of H in* $\mathfrak{L}$. It is a symmetric and completely continuous operator (in $\mathfrak{L}$). If H is positive, L is positive too.

The eigenvalues, eigenvectors, resolvent operator, etc., of L (in $\mathfrak{L}$) will be called the eigenvalues, eigenvectors, resolvent operator, etc., *of H in* $\mathfrak{L}$.

An eigenvector u and the corresponding eigenvalue λ of H in $\mathfrak{L}$ satisfy the equation

$$Hu - \lambda u = p \qquad \text{with } p \perp \mathfrak{L}. \tag{1}$$

The resolvent operator R_λ of H in $\mathfrak{L}$, for any $f \in \mathfrak{L}$ satisfy the equation

$$HR_\lambda f - \lambda R_\lambda f = f + p \text{ for } p \perp \mathfrak{L}. \tag{2}$$

2. The eigenvalues of H in $\mathfrak{L}$ can be defined by maximum-minimum problems for the positive eigenvalues and by minimum-maximum problems for the negative eigenvalues.[3] This is done in the following way. We define

$$\begin{aligned} Q(u) &= \frac{(Hu, u)}{(u, u)} \text{ for } u \neq 0 \\ Q(0) &= 0 \end{aligned} \tag{3}$$

and then for any set of n vectors ($n = 0, 1, 2, \ldots$), $\varphi_1, \varphi_2, \ldots \varphi_n$ of $\mathfrak{H}$ we consider the maximum (or minimum) of $Q(u)$ for all vectors $u \in \mathfrak{L}$ satisfying the conditions $(u, \varphi_k) = 0$, $k = 1, 2, \ldots, n$. This maximum (or minimum) is attained by some vector and will be denoted by $\lambda_n\{\varphi_k\}$ (or $\mu_n\{\varphi_k\}$). If we vary the n vectors φ_k, the $\lambda_n\{\varphi_k\}$ will attain its minimum λ_n which is the nth positive eigenvalue of H in $\mathfrak{L}$. In the same way, the minimum $\mu_n\{\varphi_k\}$, for varying φ_k, will attain its maximum μ_n which is the nth negative eigenvalue of H in $\mathfrak{L}$. In this way we get the positive and negative spectra of H in $\mathfrak{L}$:

$$\begin{aligned} &\lambda_0 \geqslant \lambda_1 \geqslant \ldots \to 0 && \text{positive spectrum,} \\ &\mu_0 \leqslant \mu_1 \leqslant \ldots \to 0 && \text{negative spectrum.} \end{aligned} \tag{4}$$

In this definition we do not exclude the vector $u = 0$. This fact makes it necessary to accept the following convention. If there are only a finite number of positive (or negative) eigenvalues, then the positive (or negative) spectrum has to be completed by an infinite number of zeros so that the sequences (4) will always have an infinite number of terms.[4] The eigenvectors corresponding to the eigenvalues (4) will be denoted by

$$\begin{matrix} u_0^+, u_1^+, \ldots \\ u_0^-, u_1^-, \ldots . \end{matrix} \tag{5}$$

When, following our convention, we continue the positive (or negative) spectrum by zeros, the corresponding u_n^+ or u_n^- will be equal to zero. All the other eigenvectors will be supposed normalized ($\|u_m^+\| = \|u_n^-\| = 1$). The operator H may admit in $\mathfrak{L}$ the eigenvalue 0. A corresponding complete system of orthonormal eigenvectors will be denoted by

$$u_1^0, u_2^0, u_3^0, \ldots . \tag{5'}$$

3. The following two theorems are well known.[5]

THEOREM A. *If $\mathfrak{L} \subset \mathfrak{L}'$ then for the corresponding eigenvalues we have*

$$\lambda_k \leqslant \lambda_k', \mu_k \geqslant \mu_k', k = 0, 1, 2, \ldots .$$

THEOREM B. *If $\mathfrak{L} \subset \mathfrak{L}'$ and $\mathfrak{L}' \ominus \mathfrak{L}$ is n-dimensional, then*

$$\lambda_k' \geqslant \lambda_k \geqslant \lambda_{k+n}', \mu_k' \leqslant \mu_k \leqslant \mu_{k+n}', k = 0, 1, \ldots .$$

The following theorem seems new even in the case of an n-dimensional space, $n < \infty$.

THEOREM I. *If $\mathfrak{L}' \subset \mathfrak{L}$, $\mathfrak{L}'' = \mathfrak{L} \ominus \mathfrak{L}'$, we have the inequalities:*

$$\lambda_{i+j} + \mu_0 \leqslant \lambda_i' + \lambda_j'', \mu_{i+j} + \lambda_0 \geqslant \mu_i' + \mu_j'', \text{ for } i, j = 0, 1, 2, \ldots .$$

An interesting special case of this theorem is

COROLLARY I'. *Under the assumptions of Theorem I, if H is positive definite we have $\lambda_{i+j} \leqslant \lambda_i' + \lambda_j''$.*

A sequence of subspaces $\mathfrak{L}^{(n)}$ is said to converge to a subspace $\mathfrak{L}$ if for every $u \in \mathfrak{H}$, $P^{(n)} u \to Pu$. We then have the theorem

THEOREM II. *If $\mathfrak{L}^{(n)} \to \mathfrak{L}$ then $\lambda_k^{(n)} \to \lambda_k$, $\mu_k^{(n)} \to \mu_k$.*

If $\mathfrak{L}^{(n)}$ is an increasing (or decreasing) sequence of subspaces, then it converges to the smallest closed subspace $\mathfrak{L}$ containing all of them (or the intersection $\mathfrak{L}$ of all of them). It follows from theorems A and II that

COROLLARY II'. *If $\mathfrak{L}^{(n)} \nearrow \mathfrak{L}$ (or $\mathfrak{L}^{(n)} \searrow \mathfrak{L}$), $\lambda_k^{(n)} \nearrow \lambda_k$, $\mu_k^{(n)} \searrow \mu_k$ (or $\lambda_k^{(n)} \searrow \lambda_k$, $\mu_k^{(n)} \nearrow \mu_k$).*

4. Consider two subspaces, $\mathfrak{L}' \subset \mathfrak{L}$ with $\mathfrak{L} \ominus \mathfrak{L}'$ n-dimensional. Consider further, a system of n vectors $p_1, p_2, \ldots, p_n$ generating the subspace $\mathfrak{L} \ominus \mathfrak{L}'$, the resolvent operator R_λ of H in $\mathfrak{L}$ and

$$u_m(\lambda) = R_\lambda p_m, \qquad m = 1, 2, \ldots, n, \qquad \lambda \text{ any complex number.} \tag{6}$$

We will denote by $W(\zeta)$ (Weinstein's determinant) the nth order determinant

$$W(\zeta) = \det. \{(u_m(\zeta), p_k)\} \tag{7}$$

and by $\Gamma \equiv \Gamma\{p_k\}$, the Gramm's determinant

$$\Gamma\{p_k\} = \det. \{(p_m, p_k)\}. \tag{8}$$

We have then the following fundamental theorem.

THEOREM III. *The function* $\Phi(\zeta) = \dfrac{W(\zeta)}{\Gamma\{p_k\}}$ *is dependent only on the operator* H, *and the subspaces* $\mathfrak{L}$ *and* $\mathfrak{L}'$. $\Phi(\zeta)$ *is meromorphic in the whole plane* ζ *with the exception of* $\zeta = 0$ (*where it may have an essential singularity*), *is regular at* $\zeta = \infty$ *and representable by the product*

$$\Phi(\zeta) = \left(\frac{-1}{\zeta}\right)^n \prod_{k=0}^{\infty} \frac{(\zeta - \lambda_k')}{(\zeta - \lambda_k)} \prod_{k=0}^{\infty} \frac{(\zeta - \mu_k')}{(\zeta - \mu_k)}, \tag{9}$$

where λ_k, μ_k, λ_k', μ_k' *are the eigenvalues of* H *in* $\mathfrak{L}$ *and* $\mathfrak{L}'$, *respectively.*

Consider now the projection P' on $\mathfrak{L}'$ and the resolvent operator R_λ' of H in $\mathfrak{L}'$. We write

$$w_k(\lambda) = -R_\lambda' P' H p_k. \tag{10}$$

Denoting by $D(\zeta)$ the determinant of nth order

$$D(\zeta) = \det. \{(Hw_k + Hp_k - \zeta p_k, p_i)\}, \tag{11}$$

We have

THEOREM IV. *The function* $\Psi(\zeta) = \dfrac{D(\zeta)}{\Gamma\{p_k\}}$ *is given by*

$$\Psi(\zeta) = \frac{1}{\Phi(\zeta)} = (-\zeta)^n \prod_{k=0}^{\infty} \frac{(\zeta - \lambda_k)}{(\zeta - \lambda_k')} \prod_{k=0}^{\infty} \frac{(\zeta - \mu_k)}{(\zeta - \mu_k')}. \tag{12}$$

5. The proof of theorems III and IV is based on a few lemmas. We use first

LEMMA 1. *The function* $\Phi(\zeta)$ *of* (9) *does not depend on the choice of vectors* $\{p_k\}$ *generating* $\mathfrak{L} \ominus \mathfrak{L}'$.

This lemma allows us to limit ourselves to the case of orthonormal vectors p_k. In writing

$$\Phi(\zeta) \equiv \Phi_{\mathfrak{L}', \mathfrak{L}}(\zeta), \tag{13}$$

we can state

LEMMA 2. *If* $\mathfrak{L}' \subset \mathfrak{L}'' \subset \mathfrak{L}$, *then* $\Phi_{\mathfrak{L}', \mathfrak{L}''} \Phi_{\mathfrak{L}'', \mathfrak{L}} = \Phi_{\mathfrak{L}', \mathfrak{L}}$.

This lemma is proved firstly in the case when $\mathfrak{L} \ominus \mathfrak{L}''$ is one dimensional and the general case follows immediately.

Following these two lemmas, the proof of Theorem I is reduced to the case when $\mathfrak{L} \ominus \mathfrak{L}'$ is one dimensional. In this case we have only one vector, p_1, $\| p_1 \| = 1$, and using the spectral representation of the resolvent operator, we obtain

$$\Phi(\zeta) = \sum_{k=0}^{\infty} \frac{|\alpha_k^+|^2}{\lambda_k - \zeta} + \sum_{k=0}^{\infty} \frac{|\alpha_k^-|^2}{\mu_k - \zeta} - \frac{1}{\zeta} \sum_{k=1}^{\infty} |\alpha_k^0|^2, \tag{14}$$

where $\alpha_k^+ = (p_1, u_k^+)$, $\alpha_k^- = (p_1, u_k^-)$, $\alpha_k^0 = (p_1, u_k^0)$.

We use then the following lemma on analytic functions.

LEMMA 3. *Any meromorphic function of the form*

$$\Phi(\zeta) = \frac{-\gamma}{\zeta} + \sum_{k=0}^{\infty} \frac{\gamma_k^+}{\omega_k^+ - \zeta} + \sum_{k=0}^{\infty} \frac{\gamma_k^-}{\omega_k^- - \zeta}, \tag{15}$$

where $\gamma_k^+ > 0$, $\gamma_k^- > 0$, $\gamma \geqslant 0$, $\gamma + \sum(\gamma_k^+ + \gamma_k^-) = 1$, $\omega_0^+ > \omega_1^+ > \ldots \to 0$ $\omega_0^- < \omega_1^- < \ldots \to 0$, *is representable in the form*

$$\Phi(\zeta) = \frac{-1}{\zeta} \prod_0^{\infty} \frac{\zeta - \tau_k^+}{\zeta - \omega_k^+} \prod_0^{\infty} \frac{\zeta - \tau_k^-}{\zeta - \omega_k^-}, \tag{16}$$

where $\omega_k^+ > \tau_k^+ > \omega_{k+1}^+$, $\omega_k^- < \tau_k^- < \omega_{k+1}^-$. *Inversely, every function of the form (16) is representable in the form (15). Both representations are unique, the* τ_k^+ *and* τ_k^- *are the zeros of* Φ *and we have the formulas:*

$$\gamma = \prod \frac{\tau_k^+}{\omega_k^+} \prod \frac{\tau_k^-}{\omega_k^-},$$

$$\gamma_l^+ = \frac{\omega_l^+ - \tau_l^+}{\omega_l^+} \prod_{k \neq l} \frac{\omega_l^+ - \tau_k^+}{\omega_l^+ - \omega_k^+} \prod \frac{\omega_l^+ - \tau_k^-}{\omega_l^+ - \omega_k^-},$$

$$\gamma_l^- = \frac{\omega_l^- - \tau_l^-}{\omega_l^-} \prod \frac{\omega_l^- - \tau_k^+}{\omega_l^- - \omega_k^+} \prod_{k \neq l} \frac{\omega_l^- - \tau_k^-}{\omega_l^- - \omega_k^-}.$$

This lemma gives us the representation (16), similar to (9) for the case of $n = 1$. In order to obtain the exact formula (9) from (16) (for $n = 1$) we use the following lemma.

LEMMA 4. *In the case of a one-dimensional* $\mathfrak{L} \ominus \mathfrak{L}'$, *for any positive (or negative) number* ζ', *the difference of multiplicity of* ζ' *in* $\{\lambda_k'\}$ *and* $\{\lambda_k\}$ *(or* $\{\mu_k'\}$ *and* $\{\mu_k\}$*) is equal to* -1, 0 *or* 1, *depending on* ζ' *being a pole, an ordinary point or a zero of* $\Phi(\zeta)$.

The term "multiplicity of ζ'" in a monotonic sequence $\{\rho_k\}$ means the number of times ζ' appears in this sequence.

After having proved Theorem III, we pass to Theorem IV. It is sufficient to prove here that $\Psi = 1/\Phi$. This is achieved by comparing the expressions of the determinants $W(\zeta)$ and $D(\zeta)$.

6. The problem in which we are interested is the calculation of the eigenvalues of the operator H in a given subspace $\mathfrak{L}$. On the basis of the preceding theorems, we will develop two methods for the computation of these eigenvalues.

A. Weinstein's Method.—Suppose that for some subspace $\mathfrak{L}^{(0)} \supset \mathfrak{L}$ we know the eigenvalues and eigenvectors of H in $\mathfrak{L}^{(0)}$, and suppose further, that for a sequence of vectors, $p_1, p_2, \ldots$ of the subspace $\mathfrak{L}^{(0)} \ominus \mathfrak{L}$, we know the vectors $v_k(\lambda) = R_\lambda^{(0)} p_k$. We consider then the subspace $\mathfrak{L}^{(n)} =$

$\mathfrak{L}^{(0)} \ominus \mathcal{P}^{(n)}$ where $\mathcal{P}^{(n)}$ is the n-dimensional subspace generated by the n first vectors p_k.

Our assumptions allow us to form explicitly the function $\Phi_n(\zeta) \equiv \Phi_{\mathfrak{L}^{(n)}, \mathfrak{L}^{(0)}}(\zeta)$, and by Theorem III, knowing the eigenvalues $\{\lambda_k^{(0)}\}$ and $\{\mu_k^{(0)}\}$ we can find the eigenvalues $\{\lambda_k^{(n)}\}$ and $\{\mu_k^{(n)}\}$

By Theorem A, we have

$$\lambda_k^{(n)} \geqslant \lambda_k^{(n+1)} \geqslant \lambda_k, \qquad \mu_k^{(n)} \leqslant \mu_k^{(n-1)} \leqslant \mu_k.$$

If the sequence $\{p_k\}$ is complete in the subspace $\mathfrak{L}^{(0)} \ominus \mathfrak{L}$, the subspaces $\mathfrak{L}^{(n)} \searrow \mathfrak{L}$ and by Corollary II', $\lambda_k^{(n)} \searrow \lambda_k$ and $\mu_k^{(n)} \nearrow \mu_k$.

Remark: We can give here a practical rule for establishing the sequence $\{\lambda_k^{(n)}\}$ (for the $\mu_k^{(n)}$ there is a similar rule). Consider the sequence of positive poles $\omega_1 \geqslant \omega_2 \geqslant \ldots$ and of positive zeros $\tau_1 \geqslant \tau_2 \geqslant \ldots$ of $\Phi_n(\zeta)$. $\{\omega_k\}$ is a subsequence of the sequence $\{\lambda_k^{(0)}\}$, i.e., it is of the form $\{\lambda_{k_l}^{(0)}\}$, $l = 1, 2, \ldots$. Consider then the complementary terms $\lambda_{k'}^{(0)}$ for all $k' \neq k_l$ and order all these terms together with the zeros τ_k in one non-increasing sequence. The obtained sequence will be $\{\lambda_k^{(n)}\}$.

B. Rayleigh-Ritz Method.—Suppose now that for some subspace $\mathfrak{L}^{(0)} \subset \mathfrak{L}$ we know completely the eigenvalues, eigenvectors, resolvent operator and the part of H in $\mathfrak{L}^{(0)}$.

Consider then a sequence of vectors $\{p_k\}$ contained in $\mathfrak{L} \ominus \mathfrak{L}^{(0)}$. We introduce the increasing sequence of subspaces $\mathfrak{L}^{(n)} = \mathfrak{L}^{(0)} \oplus \mathcal{P}^{(n)}$, where $\mathcal{P}^{(n)}$ is the n-dimensional subspace generated by the n first p_k. By our assumptions we can calculate the function $\Psi_n(\zeta) \equiv \Psi_{\mathfrak{L}^{(n)}, \mathfrak{L}^{(0)}}(\zeta)$, and knowing the eigenvalues of H in $\mathfrak{L}^{(0)}$, we obtain, by Theorem IV, the eigenvalues of H in $\mathfrak{L}^{(n)}$. By Theorem A, we have

$$\lambda_k^{(n)} \leqslant \lambda_k^{(n+1)} \leqslant \lambda_k, \; \mu_k^{(n)} \geqslant \mu_k^{(n+1)} \geqslant \mu_k.$$

If the p_k form a complete system in $\mathfrak{L} \ominus \mathfrak{L}^{(0)}$, we get $\lambda_k^{(n)} \nearrow \lambda_k$, and $\mu_k^{(n)} \searrow \mu_k$. This method will be called the *generalized Rayleigh-Ritz method.* The *ordinary Rayleigh-Ritz method* corresponds to the case when we take $\mathfrak{L}^{(0)} = (0)$, i.e., the subspace containing only the zero vector. It is clear that in this case all our assumptions will be fulfilled so that the ordinary Rayleigh-Ritz method may always be applied.

7. It is clear now that the two methods are complementary in character. The Rayleigh-Ritz method gives lower bounds for λ_k, upper bounds for μ_k, whereas the Weinstein method gives upper bounds for λ_k, lower bounds for μ_k. The combination of the two (even by the use of a small number of p_k) may lead to quite good approximations of the eigenvalues of H in $\mathfrak{L}$. A method may be developed for evaluating the errors made in taking $\lambda_k^{(n)}$ instead of λ_k. In the case of a definite positive operator we can use, for instance, Corollary I' of § 3 and obtain inequalities of the type $|\lambda_k^{(n)} - \lambda_k| \leqslant \lambda_0^{\{n\}}$, where $\lambda_k^{\{n\}}$ are the eigenvalues of H in the subspace $\mathfrak{L}^{(n)} \ominus \mathfrak{L}$

for the Weinstein method and $\mathfrak{L} \ominus \mathfrak{L}^{(n)}$ for the Rayleigh-Ritz method, respectively.

8. Lemma 3 of § 5 allows the complete solution of the following problem. Given the eigenvalues $\{\lambda_k\}$ and $\{\mu_k\}$ of H in $\mathfrak{L}$, find the characteristic properties of the eigenvalues $\{\lambda'_k\}$ and $\{\mu'_k\}$ of H in a subspace $\mathfrak{L}' \subset \mathfrak{L}$.

We will mention here only the solution of this problem in the case of a positive definite H. We have then only the sequence $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \to 0$. For $\lambda'_0 \geqslant \lambda'_1 \geqslant \lambda'_2 \geqslant \ldots \to 0$ to be the sequence of the eigenvalues of H in a subspace $\mathfrak{L}' \subset \mathfrak{L}$, it is necessary and sufficient that $\lambda'_k \leqslant \lambda_k$ for every k and that for some $h = 1, 2, \ldots$, the following condition be fulfilled

$$\prod_{n=0}^{\infty} \frac{\lambda'_{n,h}}{\lambda_n} = 0, \qquad (17_h)$$

where $\lambda'_{n,h} = \max. (\lambda'_n, \lambda_{n+h})$.

It should be remarked that for two sequences $\{\lambda_k\}$ and $\{\lambda'_k\} \searrow 0$, satisfying the condition $\lambda'_k \leqslant \lambda_k$, the conditions (17_h), for any values of h, are always equivalent. From (17_h) follows the condition

$$\prod_{0}^{\infty} \frac{\lambda'_m}{\lambda_m} = 0. \qquad (18)$$

We can get more precise information about λ'_k if we require them to be the eigenvalues of H in a subspace $\mathfrak{L}' \subset \mathfrak{L}$ such that $\mathfrak{L} \ominus \mathfrak{L}'$ be n-dimensional, $n < \infty$. Then the necessary and sufficient conditions are $\lambda_k \geqslant \lambda'_k \geqslant \lambda_{k+n}$, together with the condition (18).

* The results of the present and the following papers were obtained by the author in 1943 as far as the positive definite operators are concerned. A résumé of these results was circulated in a few copies among a number of mathematicians, but were never published. The extension to indefinite operators is much more recent.

[1] Ritz, W., *J. Reine Angew. Math.*, **135**, 1 (1908); Courant, R. *Bull. Am. Math. Soc.* **49**, 1 (1943).

[2] Weinstein, A., *Memorial des Sciences Math.*, **88** (1937).

[3] Courant-Hilbert, *Methoden der Math. Physik*, Vol. I, 2nd ed., Berlin, pp. 26–29, 112–113, 351.

[4] This kind of convention was first introduced by H. Weyl, *Math. Ann.*, **71**, 443 (footnote), 1912.

[5] These theorems are known especially for differential operators, cf. Courant-Hilbert, *loc. cit.*, p. 353. These operators in usual cases are the inverses of completely continuous integral operators and consequently the inequalities in the theorems have to be inverted.

EXPONENTIAL TRANSFORMS AND APPELL POLYNOMIALS

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1. By an exponential transform we mean an integral of the form

$$\int_0^\infty f(t)e^{\alpha zt}\,dt,$$

where α is a complex number of absolute value one. The choice $\alpha = -1$ gives the usual form of the Laplace transform. The bilateral Laplace transform is the sum of two exponential transforms with $\alpha = -1$ and $\alpha = 1$; the Fourier transform is the sum of two exponential transforms with $\alpha = i$ and $\alpha = -i$. A bilateral Laplace transform converges, in general, in a vertical strip of the complex plane; a Fourier transform converges, in general, in a horizontal strip; but not every function analytic in a vertical or horizontal strip is represented by one of these transforms. The sum of a bilateral Laplace and a Fourier transform converges, in general, in a rectangle. This fact suggests the investigation of what analytic functions are representable by such sums of transforms or, in general, by a sum of $n \geq 3$ exponential transforms in which the α's do not lie on a straight line. The convergence region of such a sum is in general a bounded convex polygon, the intersection of n half planes; it turns out that conversely every function analytic in a convex n-gon can be represented as a sum of n exponential transforms. In particular, every analytic function has (infinitely many) representations as a sum of three exponential transforms.

The proof of this result is a simple application of Cauchy's integral formula. Let $F(z)$ be analytic in a closed bounded n-gon P whose sides are line segments L_k and represent $F(z)$ in the interior of P by Cauchy's formula:

$$F(z) = \sum_{k=1}^{n} (2\pi i)^{-1} \int_{L_k} (t - z)^{-1}F(t)\,dt.$$

Let P be the intersection of the half planes $\Re(te^{i\theta_k}) < c_k$, so that the equation of L_k is $\Re(te^{i\theta_k}) = c_k$, and let $\alpha_k = -e^{i\theta_k}$. If $\Re(-\alpha_k z) > c_k$, we have $\Re(\alpha_k(t - z)) > 0$ for t on L_k and

$$\alpha_k \int_0^\infty e^{-\alpha_k(t-z)u}\,du = (t - z)^{-1};$$

since $\left|e^{-\alpha_k(t-z)}\right| = e^{c_k - R(-\alpha_k z)}$, the integral converges absolutely and uniformly (in t) for t on L_k. Hence

$$\begin{aligned}(2\pi i)^{-1} \int_{L_k} (t - z)^{-1}F(t)\,dt &= \int_0^\infty e^{\alpha_k z u}\,du \int_{L_k} (2\pi i)^{-1}\alpha_k F(t)e^{-\alpha_k t u}\,dt \\ &= \int_0^\infty e^{\alpha_k z u} f_k(u)du,\end{aligned}$$

say. Thus

$$F(z) = \sum_{k=1}^{n} \int_0^\infty e^{\alpha_k z u} f_k(u)\, du$$

in the intersection of all the half planes $\Re(\alpha_k z) < c_k$, that is, in P.

2. The applications of the result of § 1 to Appell polynomials arise as follows. An Appell set $\{p_n(z)\}$ is defined by the formal expansion

$$A(t)e^{zt} = \sum_{n=0}^{\infty} t^n p_n(z),$$

where $A(t)$ stands for the formal power series $\sum_{n=0}^{\infty} a_n t^n$ and is called the generating function of the set $\{p_n(z)\}$. The class of functions representable by series $\sum_{n=0}^{\infty} c_n p_n(z)$ depends on properties of $A(t)$; it was found by Sheffer[1] that the appropriate class of generating functions $A(t)$ for representing analytic functions in general is the class of entire functions of exponential type. Sheffer proved that, if $A(t)$ belongs to this class, every function analytic in a sufficiently large circle with center at 0 has an Appell expansion representing it in some neighborhood of 0; if the type of $A(t)$ is zero, the expansion represents the function in its circle of regularity with center at 0, just like a power series (which corresponds to $A(t) \equiv 1$). Sheffer's results were obtained from a study of linear functional equations associated with the Appell set.

Suppose now that $F(z)$ is analytic at $z = 0$ and that we represent $F(z)$ as a sum

$$F(z) = \sum_{k=1}^{m} \int_0^\infty f_k(t) e^{\alpha_k z}\, dt \qquad (1)$$

of exponential transforms. We have

$$e^{\alpha_k z t} = \{1/A(t)\} \sum_{n=0}^{\infty} t_n \alpha_k^n p_n(z),$$

and if we substitute these expansions into (1) we obtain, formally,

$$F(z) = \sum_{n=0}^{\infty} p_n(z) \cdot \sum_{k=1}^{m} \alpha_k^n \int_0^\infty \{f_k(t)/A(\alpha_k t)\} t^n\, dt,$$

a series in the polynomials $p_k(z)$. To justify the formal process for some domain of values of z we need sufficient information about the behavior of $1/A(t)$. In general it is necessary to alter the path of integration in each exponential transform in order to avoid the zeros of $A(t)$. The more precise our knowledge of the behavior of $1/A(t)$, the sharper our expansion theorem will be. It is possible in this way to prove Sheffer's general

theorem and to obtain more precise results for special classes of generating functions. An advantage of this approach is that, since the theory of the associated functional equations is not used, the Appell expansion becomes available as a tool for studying the functional equations. Furthermore, the method works equally well for a large class of the more general polynomial sets defined by

$$A(t)e^{zt} = \sum_{n=0}^{\infty} \{B(t)\}^n p_n(z),$$

where $B(0) = 0$, $B'(0) = 1$ (Sheffer's "sets of type 0") and the results so obtained could also be applied to functional equations.

To carry out the discussion outlined here requires more information about the reciprocal of an entire function than seems to be available in the literature. The details will be given elsewhere.

[1] Sheffer, I. M., "Some Applications of Certain Polynomial Classes," *Bull. Am. Math. Soc.*, **47**, 885–898 (1941); further references are given there.

NOTES ON INTEGRATION, III

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As part of our theory of general integration begun in earlier notes,[1] we shall now establish general forms of the Fubini theorem and its extensions. Since the Fubini theorem deals with multiple and iterated integration, our notation must be modified so that at least three general integrations, attached to as many different domains, can be handled simultaneously without confusion. For our present purposes it suffices to indicate clearly the particular domain to which is attached each mathematical object under consideration. Thus the different families of functions which have to be considered on a given domain X will be denoted as $\mathfrak{E}(X)$, $\mathfrak{G}(X)$, $\mathfrak{L}(X)$, and so on. Operations on such functions will be denoted, in a slightly different fashion, as E_x, N_x, L_x, and so on; and the results of applying such operations to a particular function f will be denoted as $E_x f(x)$, $N_x f(x)$, $L_x f(x)$, and so on.[2] Furthermore it will be convenient to follow the common practice in shortening the precise phrase "the function f whose value at x is $f(x)$" to the handier phrase "the function $f(x)$."

Let $Z = X \times Y$ be the Cartesian product of X and Y—that is, the totality of pairs $z = (x, y)$ where $x \in X$ and $y \in Y$. Let E_x and E_y be elementary integrals defined for the respective families $\mathfrak{E}(X)$ and $\mathfrak{E}(Y)$ of

elementary functions. We then designate by $\mathfrak{E}(X) * \mathfrak{E}(Y)$ the totality of real functions $f(z) = f(x, y)$ with the following properties: for fixed x, the function $f(x, y)$ is in $\mathfrak{E}(Y)$; and the integral $E_y f(x, y)$ is a function in $\mathfrak{E}(X)$. This family is obviously linear but is not guaranteed to contain $|f|$ whenever it contains f; it fails in this respect to conform to the requirements imposed upon a family of elementary functions. Further, we designate by $E_x * E_y$ the operation which takes any function f with the above properties into the real number $E_x E_y f(x, y)$. This is a positive linear operation which even satisfies the condition I (2) under the hypothesis that $|f|$ and $|f_n|$ are in $\mathfrak{E}(X) * \mathfrak{E}(Y)$: indeed, if $|f(x, y)| \leqq \sum_{n=1}^{\infty} |f_n(x, y)|$, application of I (2) to E_y yields $E_y|f(x, y)| \leqq \sum_{n=1}^{\infty} E_y|f_n(x, y)|$ for each x and then application of I (2) to E_x yields $E_x E_y|f(x, y)| \leqq \sum_{n=1}^{\infty} E_x E_y|f_n(x, y)|$. If $\mathfrak{E}(Z)$ is a linear subfamily of $\mathfrak{E}(X) * \mathfrak{E}(Y)$ which contains $|f|$ together with f and if E_z is the contraction of $E_x * E_y$ to $\mathfrak{E}(Z)$, we therefore see that $\mathfrak{E}(Z)$ and E_z satisfy I (1) and I (2); and hence that E_z can be regarded as an elementary integral and $\mathfrak{E}(Z)$ as the family of elementary functions over which it is defined. There are many important examples where $\mathfrak{E}(Z)$ can be specified so as to contain all the functions $h(z) = h(x, y) = f(x)g(y)$ where $f \in \mathfrak{E}(X)$ and $g \in \mathfrak{E}(Y)$, such functions h obviously being members of $\mathfrak{E}(X) * \mathfrak{E}(Y)$. Frequently E_z is given directly without reference to E_x and E_y, and has to be identified as a contraction of $E_x * E_y$—in other words, the relation $E_z \subset E_x * E_y$ has to be proved as a theorem. Since illustrations of these remarks are well known we shall not go into greater detail here. We must, however, call particular attention to the fact that in general we will have $\mathfrak{E}(X) * \mathfrak{E}(Y) \neq \mathfrak{E}(Y) * \mathfrak{E}(X)$ and $E_x * E_y \neq E_y * E_x$. This lack of symmetry may well extend so far that for some functions f we have $E_x E_y f(x, y) \neq E_y E_x f(x, y)$; but, of course, the functions $h(z) = h(x, y) = f(x)g(y)$ where $f \in \mathfrak{E}(X)$ and $g \in \mathfrak{E}(Y)$ are not among them. On the other hand there are many familiar and important cases where $\mathfrak{E}(Z)$ can be specified so that $E_x * E_y$ and $E_y * E_x$ have identical contractions to $\mathfrak{E}(Z)$. In such a case we have $E_z f(z) = E_x E_y f(x, y) = E_y E_x f(x, y)$ and the relation of E_z to E_x and E_y involves the latter in a symmetric manner.

Turning now to the general integrations L_x and L_y associated with E_x and E_y, respectively, we shall introduce an operation $L_x * L_y$ analogous to the operation $E_x * E_y$ of the preceding paragraph. First we designate by $\mathfrak{L}(X) * \mathfrak{L}(Y)$ the family of all extended-real functions $f(z) = f(x, y)$ with the following properties: for each x outside a fixed null subset X_0 of X the function $f(x, y)$ is in $\mathfrak{L}(Y)$; and the integral $L_y f(x, y)$ is a function defined outside X_0 and coinciding there with a function g in $\mathfrak{L}(X)$. We then define $L_x * L_y$ as the operation which takes such a function f into the

real number $L_xg(x)$ where $g(x) = L_yf(x, y)$ for x outside X_0 and $g \in \mathfrak{L}(X)$, observing that this number is unaffected by the ambiguity in the determination of g. As we shall see below, the generalized Fubini theorem is conveniently expressed in terms of the operation $L_x * L_y$.

As a preliminary to the statement and proof of the generalized Fubini theorem, we first make the following observation:

(1) *if the elementary integrations E_x, E_y, and E_z satisfy the relation $E_z \subset E_x * E_y$, then the corresponding operations N_x, N_y, N_z are such that $N_zf(z) \geqq N_xN_yf(x, y)$ for every f in $\mathfrak{G}(Z)$.*

The proof is simple. Since we have nothing to prove unless $N_zf(z) < +\infty$, we assume the latter relation. We can then choose f_n in $\mathfrak{E}(Z)$ so that $|f| \leqq \sum_{n=1}^{\infty} |f_n|$ and $\sum_{n=1}^{\infty} E_xE_y|f_n(x_n, y)| = \sum_{n=1}^{\infty} E_z|f_n(z)| \leqq N_zf(z) + \epsilon$ for any given $\epsilon > 0$. On the other hand we have $N_yf(x, y) \leqq \sum_{n=1}^{\infty} N_yf_n(x, y) = \sum_{n=1}^{\infty} E_y|f_n(x, y)|$ by I (7) and I (9); and hence $N_xN_yf(x, y) \leqq \sum_{n=1}^{\infty} N_xE_y|f_n(x, y)| = \sum_{n=1}^{\infty} E_xE_y|f_n(x, y)| \leqq N_zf(z) + \epsilon$ by I (7), I (9), and the above. Since $\epsilon > 0$ is arbitrary, the theorem is established.

We now state the first part of the generalized theorem of Fubini:

(2) (Fubini) *if the elementary integrations E_x, E_y, and E_z satisfy the relation $E_z \subset E_x * E_y$, then the corresponding general integrations satisfy the relation $L_z \subset L_x * L_y$—in other words the general integral $L_zf(z)$ can be evaluated as the iterated integral $L_xL_yf(x, y)$ in the sense made precise above.*

With the help of (1) the proof offers little difficulty. If f is in $\mathfrak{L}(Z)$ we can find f_n in $\mathfrak{E}(Z)$ so that $N_z(f(z) - f_n(z)) \leqq \cdot 2^{-n}$. The positive-term series $\sum_{n=1}^{\infty} N_y(f(x, y) - f_n(x, y))$ has sum $h(x)$ in $\mathfrak{G}(X)$; and the relation $N_xh(x) \leqq \sum_{n=1}^{\infty} N_xN_y(f(x, y) - f_n(x, y)) \leqq \sum_{n=1}^{\infty} N_z(f(z) - f_n(z)) \leqq \sum_{n=1}^{\infty} 2^{-n} = 1$ shows that $h \in \mathfrak{F}(X)$ and hence that h is finite except on a null set X_0. If x is outside X_0 we therefore have $\lim_{n\to\infty} N_y(f(x, y) - f_n(x, y)) = 0$ and hence $f(x, y) \in \mathfrak{L}(Y)$. We let $g(x)$ be any function in $\mathfrak{G}(X)$ which is equal to $L_yf(x, y)$ outside X_0. Since $|g(x) - E_yf_n(x, y)| = |L_yf(x, y) - E_yf_n(x, y)| = |L_y(f(x, y) - f_n(x, y))| \leqq L_y|f(x, y) - f_n(x, y)| = N_y(f(x, y) - f_n(x, y))$ almost everywhere, we see that $N_x(g(x) - E_yf_n(x, y)) \leqq N_xN_y(f(x, y) - f_n(x, y)) \leqq N_z(f(z) - f_n(z)) \leqq 2^{-n}$. Hence $g(x)$ is in $\mathfrak{L}(X)$ and $L_zg(x) = \lim_{n\to\infty} E_xE_yf_n(x, y) = \lim_{n\to\infty} E_zf_n(z) = L_zf(z)$ in accordance with the definitions of L_x and L_z. The remainder of the generalized theorem of Fubini is the following partial converse of (2):

(3) (Fubini) *if the elementary integrations* E_x, E_y, *and* E_z *satisfy the relation* $E_z \subset E_x * E_y$, *if* $f \in \mathfrak{M}(Z)$, *and if there exist functions* f_n *in* $\mathfrak{F}(Z)$ *such that* $|f| \leqq \sum_{n=1}^{\infty} |f_n|$, *then* $|f| \in \mathfrak{L}(X) * \mathfrak{L}(Y)$ *implies* $f \in \mathfrak{L}(Z)$.

In accordance with II (14) we may suppose without loss of generality that $f_n \in \mathfrak{L}(Z)$. The function $g_n = \min(|f|, |f_1| + \ldots + |f_n|)$ is also in $\mathfrak{L}(Z)$ in accordance with II (6), II (7). Since $0 \leqq g_n \leqq |f|$ we see that $N_y g_n(x, y) \leqq N_y f(x, y)$ and $N_x N_y g_n(x, y) \leqq N_x N_y f(x, y)$. The fact that $|f|$ is in $\mathfrak{L}(X) * \mathfrak{L}(Y)$ shows that $N_y f(x, y) = L_y|f(x, y)|$ for almost all x, and also that $N_y f(x, y)$ differs only on a null set from an integrable function $g(x)$ and is therefore integrable itself. Thus $N_x N_y f(x, y) = L_x g(x) < +\infty$. On the other hand the inequalities $g_n \leqq g_{n+1}$ and $|f| \leqq \sum_{n=1}^{\infty} |f_n|$ show that $\{g_n\}$ is a monotonely increasing sequence which converges to $|f|$. Hence $N_x f(z) = \lim_{n \to \infty} L_x g_n(z)$. Application of (2) to $g_n \in \mathfrak{L}(Z)$ yields $L_x g_n(z) = N_x N_y g_n(x, y) \leqq N_x N_y f(x, y)$. Thus $N_x f(z) \leqq N_x N_y f(x, y) < +\infty$ so that $f \in \mathfrak{F}(Z)$. Since it was assumed that $f \in \mathfrak{M}(Z)$, we conclude by II (11) that $f \in \mathfrak{L}(Z)$. It is well known that the hypotheses of (3) cannot be weakened in any essential respect. Membership in $\mathfrak{L}(X) * \mathfrak{L}(Y)$ does not imply membership in $\mathfrak{M}(Z)$, so that some hypothesis concerning the measurability of f is needed in order to guarantee that $f \in \mathfrak{L}(Z)$. The need for the condition that $|f| \leqq \sum_{n=1}^{\infty} |f_n|$ for appropriate f_n in $\mathfrak{F}(Z)$ is illustrated by a simple example of Saks.[3] This condition is automatically satisfied in many of the common instances of our general theory. In particular if $1 \in \mathfrak{L}(Z)$ we can always take $f_n = 1$.

Our version of the Fubini theorem can be applied directly to a situation in the theory of locally compact topological groups.[4] Let Z be such a group, Y one of its closed subgroups and X the homogeneous space of left cosets of Y. Selecting from each coset x a fixed element z_x we see that the equation $z = z_x y$ defines a one-to-one correspondence between Z and $X \times Y$, which can therefore be identified as abstract sets during the remainder of the discussion. We let $\mathfrak{E}(X)$, $\mathfrak{E}(Y)$, $\mathfrak{E}(Z)$ be the families of continuous real functions with compact nuclei on the respective spaces X, Y, Z. Three elementary integrations E_x, E_y, E_z defined over $\mathfrak{E}(X)$, $\mathfrak{E}(Y)$, $\mathfrak{E}(Z)$, respectively, will be said to form an admissible triple if E_y is left-invariant, E_x and E_z are relatively left-invariant and $E_z f(z) = E_x E_y f(z'y)$ for all f in $\mathfrak{E}(Z)$. It is implicit in this definition that the integral $E_y f(z'y)$ is constant on each coset x and can therefore be considered as a function on X which is, in fact, a member of $\mathfrak{E}(X)$. We observe now that E_x, E_y, E_z constitute an admissible triple if and only if, in addition to

enjoying the required invariance properties, they satisfy the relation $E_z \subset E_x * E_y$. To verify this we need only note that for any f in $\mathfrak{E}(Z)$ we have $f(z) = f(z_x y) = f(x, y)$ and hence $E_y f(x, y) = E_y f(z_x y) = E_y f(z'y)$ for all z' in the coset x: for we then have $E_y f(x, y) \in \mathfrak{E}(X)$ and the conditions $E_z f(z) = E_x E_y f(z'y)$ and $E_z \subset E_x * E_y$ are therefore equivalent. The group-theoretic conditions for the existence of admissible triples are discussed by A. Weil.[4] Here we direct attention to the fact that when $\{E_x, E_y, E_z\}$ is an admissible triple, *the associated general integrations* L_x, L_y, L_z (which enjoy corresponding invariance properties) *must satisfy the relation* $L_z \subset L_x * L_y$ *in accordance with* (2) *above.*[5]

We turn finally to an extension of the Fubini theorem, due originally to Jessen in a particular case.[6] With each element λ of a fixed infinite class Λ let there be associated a non-void set X_λ. We denote by X_Λ the Cartesian product of those X_λ with $\lambda \in \Lambda$. For each non-void *finite* part B of Λ let there be given an elementary integration E_{x_B} defined for a family $\mathfrak{E}(X_B)$ of elementary functions on X_B. Let it be assumed that the following conditions hold:

(4) *the constant function everywhere equal to* 1 *on* X_B *is in* $\mathfrak{E}(X_B)$ *and its elementary integral is* 1;

(5) *if* Γ *and* Δ *constitute a partition of* B *and if* $g \in \mathfrak{E}(X_\Gamma)$, *then the function* f *defined by* $f(x_B) = f(x_\Gamma, x_\Delta) = g(x_\Gamma)$ *is in* $\mathfrak{E}(X_B)$;

(6) *if* Γ *and* Δ *constitute a partition of* B, *then* $E_{x_B} \subset E_{x_\Gamma} * E_{x_\Delta}$.

For an arbitrary infinite part A of Λ we can now define[7] $\mathfrak{E}(X_A)$ and E_{x_A} satisfying I (1): we take $\mathfrak{E}(X_A)$ to be the family of those functions f on X_A such that for some partition of Λ into a finite set B and its complement Γ and for some g in $\mathfrak{E}(X_B)$ the relation $f(x_A) = f(x_B, x_\Gamma) = g(x_B)$ is valid; and for each such f we put $E_{x_A} f(x_A) = E_{x_B} f(x_B, x_\Gamma) = E_{x_B} g(x_B)$. We assume finally that I (2) holds for $\mathfrak{E}(X_A)$ and E_{x_A}. It is then evident that I (2) must also hold for $\mathfrak{E}(X_A)$ and E_{x_A}, whatever the infinite set $A \subset \Lambda$. As an instance where all our assumptions can easily be verified, we cite one equivalent to that given by Jessen:[6] we take X_λ to be the unit interval, $0 \leqq x_\lambda \leqq 1$; $\mathfrak{E}(X_B)$ to be the family of all continuous real functions on the hypercube X_B; and E_{x_B} to be the Riemann integral. In the general case a rather simple analysis, which we shall not repeat, shows that

(7) *in* (4), (5) *and* (6) *the finite set* B *can be replaced by an arbitrary infinite set* $A \subset \Lambda$.

We now direct attention to a remarkable property of the general integration associated with E_{x_A}, namely:

(8) *if* $f \in \mathfrak{L}_p(X_A)$ *then there exist a partition of* Λ *into sets* Γ *and* Δ, *where* Γ *is countable, and a function* g *in* $\mathfrak{L}_p(X_\Gamma)$ *such that* $f(x_A) = f(x_\Gamma, x_\Delta) = g(x_\Gamma)$ *for almost all* x_A.

Because of the mapping of $\mathfrak{L}_p(X_A)$ onto $\mathfrak{L}(X_A)$ discussed in II, it suffices to treat the case $p = 1$. There we determine functions f_n in $\mathfrak{E}(X_A)$ such that $N(f - f_n) \leqq 2^{-n}$. As in the proof of I (10), we find that f and $\limsup_{n\to\infty} f_n = h$ differ only on a null set. In view of the definition of $\mathfrak{E}(X_A)$ there exists a countable set Γ such that for every n the function $f_n(x_A) = f_n(x_\Gamma, x_\Delta)$ is constant with respect to x_Δ. Hence $h(x_A) = h(x_\Gamma, x_\Delta)$ has the same property. The function $g(x_\Gamma) = h(x_\Gamma, x_\Delta)$ is then in $\mathfrak{L}(X_\Gamma)$, since (6) and (7) above permit the application of (2). Jessen's most interesting results concern the case where A is countable; but (8) shows that in handling a finite or countably infinite family of functions in $\mathfrak{L}_p(X_A)$ there is no loss of generality in restricting attention to this case. We suppose therefore that A is the class of positive integers, and establish the following result:

(9) (Jessen) *if* $f \in \mathfrak{L}_p(X_A)$, $B = \{\alpha;\ \alpha \leqq n\}$, *and* $\Gamma = \{\alpha;\ \alpha > n\}$, *then there exist functions* g_n, h_n *in* $\mathfrak{L}_p(X_A)$ *defined for almost all* X_A *by the relations* $g_n(x_A) = g_n(x_B, x_\Gamma) = L_{x_B} f(x_B, x_\Gamma)$, $h_n(x_A) = h_n(x_B, x_\Gamma) = L_{x_\Gamma} f(x_B, x_\Gamma)$; *and in* $\mathfrak{L}_p(X_A)$ *the convergence relations* $g_n \to L_{x_A} f(x_A)$, $h_n \to f$ *are valid.*

Only h_n will be discussed in detail, as g_n can be treated in much the same way. Since $1 \in \mathfrak{L}(X_\Delta)$ for any $\Delta \subset A$, we see that $\mathfrak{L}_p(X_\Delta) \subset \mathfrak{L}(X_\Delta)$ and that all the techniques developed in II are available to us here. The Fubini theorem is also available to us. Thus we can use this theorem to infer first that h_n exists as a member of $\mathfrak{L}(X_A)$. If f is in $\mathfrak{L}_p(X_A)$, the inequality $N_{x_A}|h_n(x_A)|^p = N_{x_B}|L_{x_\Gamma} f(x_B, x_\Gamma)|^p \leqq N_{x_B} L_{x_\Gamma} |f(x_B, x_\Gamma)|^p = L_{x_B} L_{x_\Gamma} |f(x_B, x_\Gamma)|^p = L_{x_A} |f(x_A)|^p = N_{x_A} |f(x_A)|^p < +\infty$ shows that h_n is also in $\mathfrak{L}_p(X_A)$. If f is in $\mathfrak{L}_p(X_A)$ we can find for any $\epsilon > 0$ a function $\bar{f}$ in $\mathfrak{L}_p(X_A)$ such that $\bar{f}(x_A) = \bar{f}(x_B, x_\Gamma)$ is constant with respect to x_Γ for some choice of $B = \{\alpha;\ \alpha \leqq m\}$ while $N_p(f - \bar{f}) \leqq \frac{1}{2}\epsilon$. Using the notations of II, we choose $\bar{f}$ so that $\bar{g} = \Phi(\bar{f}) \in \mathfrak{E}(X_A) \subset \mathfrak{L}(X_A)$ and $N(\Phi(f) - \bar{g}) \leqq \delta$. Since $B = \{\alpha;\ \alpha \leqq m\}$ can be chosen so that $\bar{g}(x_A) = \bar{g}(x_B, x_\Gamma)$ is constant with respect to x_Γ, we see that $\bar{f} = \Psi(\bar{g})$ has a like property and belongs to $\mathfrak{L}_p(X_A)$. If δ has been taken sufficiently small, it is clear that $N_p(f - \bar{f}) \leqq \frac{1}{2}\epsilon$ by virtue of the continuity of Ψ. Now if $n \geqq m$, m being the integer just determined in our choice of $\bar{f}$, we see that $\bar{h}_n(x_A) = L_{x_\Gamma} \bar{f}(x_B, x_\Gamma) = \bar{f}(x_B, x_\Gamma) = \bar{f}(x_A)$ because $\bar{f}(x_B, x_\Gamma)$ is constant with respect to x_Γ, $\Gamma = \{\alpha;\ \alpha > n\}$. Hence we have $N_p(f - h_n) = N_p(f - \bar{f} + \bar{h}_n - h_n) \leqq N_p(f - \bar{f}) + N_p(\bar{h}_n - h_n)$ for $n \geqq m$, by Minkowski's inequality. By our choice of $\bar{f}$ we have $N_p(f - \bar{f}) \leqq \frac{1}{2}\epsilon$ and $N_p(\bar{h}_n - h_n) = (L_{x_A}|\bar{h}_n(x_A) - h(x_A)|^p)^{1/p} = (L_{x_B} L_{x_\Gamma} |L_{x_\Gamma}(\bar{f}(x_B, x_\Gamma) - f(x_B, x_\Gamma))|^p)^{1/p} = (L_{x_B}|L_{x_\Gamma}(\bar{f}(x_B, x_\Gamma) - f(x_B, x_\Gamma)|^p)^{1/p} \leqq (L_{x_B} L_{x_\Gamma} |\bar{f}(x_B, x_\Gamma) - f(x_B, x_\Gamma)|^p)^{1/p} = (L_{x_A}|\bar{f}(x_A) - f(x_A)|^p)^{1/p} = N_p(f - \bar{f}) \leqq \frac{1}{2}\epsilon$. Hence $N_p(f - h_n) \leqq \epsilon$ for $n \geqq m$, as we wished to show. A further result of Jessen will complete our discussion, namely:

(10) (Jessen) *the relations* $\lim_{n\to\infty} g_n(x_A) = Lx_A f(x_A)$, $\lim_{n\to\infty} h_n(x_A) = f(x_A)$ *hold almost everywhere in the pointwise sense.*

We consider only h_n, modeling our treatment on that already given by Jessen[6] for g_n. Since $\mathfrak{L}_p(X_A) \subset \mathfrak{L}(X_A)$ we may suppose that $p = 1$ in the present instance. The sequence $\{h_n\}$, being convergent to f in $\mathfrak{L}(X_A)$ has a subsequence which converges almost everywhere to f in the pointwise sense. Consequently $h = \lim_{n\to\infty} \sup h_n \geqq f$ almost everywhere. By a method which will be sketched below we show that at almost every point of the set $X^\lambda = \{x_A;\ h(x_A) > \lambda\}$ we must have $f(x_A) \geqq \lambda$. It then follows that $\lim_{n\to\infty} \sup h_n(x_A) = f(x_A)$ almost everywhere. Replacing f by $-f$, we have to replace h_n by $-h_n$. We therefore have $\lim_{n\to\infty} \inf h_n(x_A) = -\lim_{n\to\infty} \sup (-h_n(x_A)) = -(-f(x_A)) = f(x_A)$, a relation which completes the proof of the theorem. Reverting now to the detailed study of X^λ, we let $f^\lambda_{np} \in \mathfrak{L}(X_A)$ be the characteristic function of the set $\{x_A;\ h_{n+p}(x_A) > \lambda,\ h_k(x_A) \leqq \lambda$ *for* $n \leqq k \leqq n + p - 1\}$, $n = 1, 2, 3, \ldots$ and $p = 0, 1, 2, \ldots$. The characteristic function of X^λ is then expressible as $f^\lambda = \lim_{n\to\infty} \sum_{p=0}^{\infty} f^\lambda_{np} \in \mathfrak{L}(X_A)$. Since h_k is constant with respect to x_l for $l \geqq k + 1$ we see that h_{n+p} and f_{np} are both constant with respect to x_l for $l \geqq n + p + 1$. Let g be an arbitrary function in $\mathfrak{L}(X_A)$ which is constant with respect to x_l for $l \geqq m + 1$ and which satisfies the inequalities $0 \leqq g \leqq 1$. Taking $n \geqq m$ we note that $f^\lambda_{np}g$ is constant with respect to x_l for $l \geqq n + p + 1$ and hence can be multiplied into both members of the equation $h_{n+p}(x_A) = Lx_\Gamma f(x_B, x_\Gamma)$ to yield $h_{n+p}(x_A)f^\lambda_{np}(x_A)g(x_A) = Lx_\Gamma[f(x_B, x_\Gamma)f^\lambda_{np}(x_B, x_\Gamma)g(x_B, x_\Gamma)]$. Applying $L = Lx_A$ to both members of the latter equation we obtain $L(h_{n+p}f^\lambda_{np}g) = L(ff^\lambda_{np}g)$. Since $h_{n+p} > \lambda$ on the set where $f^\lambda_{np} = 1$ it follows that $\lambda L(f^\lambda_{np}g) \leqq L(ff^\lambda_{np}g)$ and hence that $\lambda L(f^\lambda g) \leqq L(ff^\lambda g)$. It is not difficult to determine a sequence of functions g of the kind admitted here which converges boundedly to the characteristic function $g^{\lambda-\epsilon}$ of the set $\{x_A;\ f(x_A) \leqq \lambda - \epsilon\}$, exception being made as usual for points of a null set. Passage to the limit in the above inequality therefore yields $\lambda L(f^\lambda g^{\lambda-\epsilon}) \leqq L(ff^\lambda g^{\lambda-\epsilon}) \leqq (\lambda - \epsilon)L(f^\lambda g^{\lambda-\epsilon})$. Hence $\epsilon > 0$ implies $L(f^\lambda g^{\lambda-\epsilon}) = 0$; in other words the part of X^λ where $f(x_A) \leqq \lambda - \epsilon$ is a null set. Thus we must have $f(x_A) \geqq \lambda$ almost everywhere on X_A, as we claimed above.

[1] Stone, M. H., "Notes on Integration, I," these PROCEEDINGS, **34**, 336–342 (1948); "Notes on Integration, II," *Ibid.* 447–455 (1948); cited here as I and II, respectively.

[2] The symbol x in these expressions denotes a bound variable.

[3] Saks, S., *Theory of the Integral*, 2nd revised ed., Warszawa-Lwow, 1937, pp. 87–88.

[4] Weil, A., *L'Intégration dans les Groupes Topologiques et ses Applications*, Paris, 1938, pp. 30–45, especially 42–45.

[5] While this result resembles one established by Ambrose, W., "Direct Sum Theorem for Haar Measures," *Trans. Am. Math. Soc.*, **61**, 122–127 (1947), it is actually identical with the latter only in the case where Z is separable. The reason for the distinction which must be made in the non-separable case is indicated in the fourth footnote of II.

[6] Jessen, B., "The Theory of Integration in a Space of an Infinite Number of Dimensions," *Acta Mathematica*, **63**, 249–323 (1934), especially 272–280.

[7] $\mathfrak{E}(X_A)$ and $E_{\pi A}$ have in a general way the character of "projective limits" of the given $E(X_B)$ and $E_{\pi B}$, the conditions (5) and (6) being "consistency conditions" essential to the constructive process.

ERRATUM

The eleventh (last) pentad of the second line of the value for $340!/10^{83}$ on p. 409 (August) of volume 34 of these PROCEEDINGS should read 85229 in place of 58229.

HORACE S. UHLER